2.) Basics of Hamiltonian Mechanics

Liouville's Theorem

→ Basic Hamiltonian Mechanics

– Why ?

Observe L.E. : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$

$\dot{q}_i$ → generalized velocity

$\frac{\partial L}{\partial \dot{q}_i} = p_i$

p_i → generalized momentum

Lagrangian mechanics: 2^{nd} order equations for generalized coordinates

1) Then useful (for phase space descriptions) to have alternative description in terms of **first order** equations for generalized {coordinates, momenta}

→ treat coordinate and momenta on equal footing

2) Exploit energy conservation

– Hamiltonian

In general, for conservative system

$L = L(q, \dot{q})$

$dL = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q} = \frac{\partial L}{\partial q} dq + p\, d\dot{q}$

and using Lagrange's equations, $\partial L/\partial q = \frac{d}{dt} p$

seek eliminate, replace by ( ) dp (as want description in terms p, q)

$\therefore \quad dL = \dot{p}\,dq + p\,d\dot{q}$

Now, $\quad p\,d\dot{q} = d(p\dot{q}) - \dot{q}\,dp$ toward Legendre transform

$\Rightarrow \quad dL = \dot{p}\,dq + d(p\dot{q}) - \dot{q}\,dp$

$$\therefore \quad \boxed{\begin{aligned} &d(p\dot{q} - L) = -\dot{p}\,dq + \dot{q}\,dp \\ &H \equiv p\dot{q} - L \equiv \text{Hamiltonian} \end{aligned}}$$

$H = H(q, p) \equiv$ function of generalized coordinates and momenta

Now $\quad dH = -\dot{p}\,dq + \dot{q}\,dp$

but

$$dH = \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp \qquad \text{(defn.)}$$

$$\Rightarrow \quad \boxed{\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}} \qquad \text{Hamilton's Equations}$$

Notice,

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial q}\dot{q} = \frac{\partial H}{\partial t}$$

iii.) Hamiltonian description requires invertibility of generalized velocities in terms generalized momenta

i.e. see example: Hamiltonian in spherical coordinates, pg. 26

$\Rightarrow$ need solve $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$ for $\dot{q}_i(q,p)$

to eliminate. locally $d\underline{p}_i = \underline{\underline{A}} \cdot d\underline{\dot{q}}_i$

$\Rightarrow$ can re-write: $\underline{p}_i = \underline{\underline{A}} \cdot [\underline{\dot{q}}_i]$

$$\underline{\underline{A}}_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$$

invertibility/solvability $\Rightarrow$

$$\det \underline{\underline{A}} = \left| \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right| \neq 0.$$

If $\det \underline{A} = 0 \Rightarrow$ special constraint, requires treatment by Dirac brackets.

Note: i.) H not necessarily conserved/constant
i.e. can construct H even if $\partial L/\partial t \neq 0$,
E not conserved

ii.) $H = E = \text{const}$ only for closed system.

Eg. Hamiltonian in Spherical coordinates

$L = T - U$

$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$

$\therefore \quad L = (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2)\frac{m}{2} \quad - V$

$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$

$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$

$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2\theta \, \dot{\phi}$

$H = \underline{p} \cdot \underline{\dot{q}} - L$, express $\dot{q}$ as p

$\Rightarrow$

$$H = p_r\left(\frac{p_r}{m}\right) + p_\theta\left(\frac{p_\theta}{mr^2}\right) + p_\phi\left(\frac{p_\phi}{mr^2\sin^2\theta}\right)$$

$$- \left[\frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} - V\right]$$

$$H = \frac{p_r^3}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{mr^2\sin^2\theta} + V \quad \checkmark$$

Examples: Hamiltonian Mechanics

1) If $L(q,\dot{q}) = \exp[\dot{q}]$, derive the Hamiltonian?

$$H = p\dot{q} - L$$

Need eliminate $\dot{q}$ in terms p, so

$$p = \frac{\partial L}{\partial \dot{q}} = e^{\dot{q}} \quad \Rightarrow \quad \dot{q} = \ln p$$

$$\therefore \quad H = p\ln p - p$$

check: $\dot{q} = \dfrac{\partial H}{\partial p} = \ln p$

2) $L = \frac{1}{2} G(q,t)\,\dot{q}^2 + F(q,t)\,\dot{q} - V(q,t)$

a) Derive the Hamiltonian

Need eliminate $\dot{q}$, so $\Rightarrow$

$$p = \frac{\partial L}{\partial \dot{q}} = G\dot{q} + F \quad \Rightarrow \quad \dot{q} = \left(\frac{p-F}{G}\right)$$

$$H = p\dot{q} - L$$

$$= p\left(\frac{p-F}{G}\right) - \frac{1}{2}G\left(\frac{p-F}{G}\right)^2 - F\left(\frac{p-F}{G}\right) + V$$

$$= \frac{p^2}{G} - \frac{pF}{G} - \frac{G}{2}\left(\frac{p^2}{G^2} - \frac{2pF}{G^2} + \frac{F^2}{G^2}\right) - \frac{Fp}{G} + \frac{F^2}{G} + V$$

$$= \frac{p^3}{2G} - \frac{pF}{G} + \frac{F^2}{2G} + V$$

$$H = \frac{(p-F)^2}{2G} + V$$

b) Eqns. of motion, from L?

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

$$\frac{d}{dt}\left(G\dot{q} + F\right) - \frac{G'}{2}\dot{q}^2 - F'\dot{q} + \frac{\partial V}{\partial q} = 0$$

$$\Rightarrow \frac{d}{dt}\left(G\dot{q}\right) + \cancel{\frac{\partial F}{\partial q}\dot{q}} - \frac{\partial G}{\partial q}\frac{\dot{q}^2}{2} - \cancel{\frac{\partial F}{\partial q}\dot{q}} + \frac{\partial V}{\partial q} + \frac{\partial F}{\partial t} = 0$$

$$\frac{d}{dt}\left(G\dot{q}\right) - \frac{\partial G}{\partial q}\frac{\dot{q}^2}{2} + \frac{\partial V}{\partial q} + \frac{\partial F}{\partial t} = 0$$

Now, define:

$$\bar{L} = \frac{G(q,t)}{2}\dot{q}^2 - \left[V(q,t) + \frac{\partial f(q,t)}{\partial t} \right] \qquad \frac{\partial f^T}{\partial q} = F$$

i) Show Lagrange equations same.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

$$\frac{d}{dt}\left(G\dot{q}\right) - \frac{G'}{2}\frac{\dot{q}^2}{2} + \frac{\partial V}{\partial q} + \frac{\partial}{\partial q}\frac{\partial f}{\partial t} = 0$$

$\partial f / \partial q = F$

$$\frac{d}{dt}(G\dot{q}) - \frac{G'}{2}\dot{q}^2 + \frac{\partial V}{\partial q} + \frac{\partial}{\partial t} F = 0 \quad \checkmark$$

ii) Derive $\bar{H}$

$$p = \frac{\partial L}{\partial \dot{q}} = G\dot{q}$$

$$H = p\dot{q} - L$$

$$= p\left(\frac{p}{G}\right) - \frac{G}{2}\frac{p^2}{G^2} + V + \frac{\partial f}{\partial t}$$

$$H = p^2/2G + V + \partial f/\partial t \quad \checkmark$$

no explicit time dependence $(\partial L/\partial t = 0)$.

thus, if $\partial H/\partial t = 0$, $dH/dt = 0$ and energy conserved, H const.

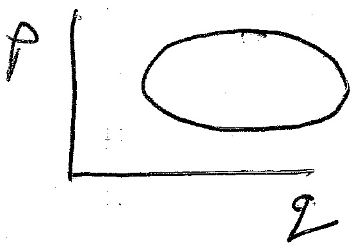
Obviously: H constructible even if energy not conserved!!

Comments re: Hamiltonians:

i.) Hamiltons equations treat p, q 'symmetrically' i.e. 2 first order ODE's in time replace 2nd order L.E.'s.

ii.) p_i, q_i are natural variables for phase space description of motion

H.O.



natural variables for phase space flow.

Why? $\Rightarrow$ For Hamiltonian system, phase space flow is incompressible.

i.e. let $\rho = \rho(q_i, p_i) \equiv$ phase space density

For particles not created or destroyed,

$$\frac{\partial \rho}{\partial t} + \sum_i \left\{ \frac{\partial}{\partial q_i}\left(\dot{q}_i \, \rho\right) \frac{\partial}{\partial p_i}\left(\dot{p}_i \, \rho\right) \right\} = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \sum_i \left\{ \left(\dot{q}_i \cdot \frac{\partial \rho}{\partial q_i} + \dot{p}_i \cdot \frac{\partial \rho}{\partial p_i} \right) + \rho \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right) \right\}$$

$$= 0$$

akin $\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho + \rho \underbrace{\nabla \cdot \underline{v}}_{\text{compression term}} = 0$

but $\dot{q}_i = \frac{\partial H}{\partial p_i}$; $\Rightarrow \frac{\partial \dot{q}_i}{\partial q_i} = \frac{\partial^2 H}{\partial q_i \partial p_i}$

$\dot{p}_i = -\frac{\partial H}{\partial q_i}$ $\Rightarrow \frac{\partial \dot{p}_i}{\partial p_i} = \frac{-\partial^2 H}{\partial q_i \partial p_i}$

so phase space flow for Hamiltonian system is incompressible!

characteristics → Hamiltonian eqns.

$$\frac{\partial \rho}{\partial t} + \sum_i \left(\dot{q}_i \cdot \frac{\partial \rho}{\partial q_i} + \dot{p}_i \cdot \frac{\partial \rho}{\partial p_i} \right) = 0$$

i.e. $\frac{d\rho}{dt} = 0$ along Hamiltonian trajectories

→ Liouville's Theorem

i.e. phase space flow incompressible
volume element conserved } in Hamiltonian system..

Review Examples

Hamilton's Eqns. / Liouville Thm.

$$H = p\dot{q} - L \quad , \qquad \dot{q} = \frac{\partial H}{\partial p} \quad ; \quad \dot{p} = -\frac{\partial H}{\partial q}$$

Liouville Thm. $\quad \underline{\nabla}_\Gamma \cdot \underline{V}_\Gamma = \frac{\partial}{\partial q}\dot{q} + \frac{\partial \dot{p}}{\partial p} = 0$

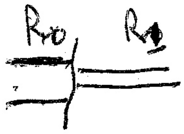
$\Downarrow$

$\left\{ \begin{array}{l} \text{phase space density conserved along traj.} \\ \text{"} \quad \text{"} \quad \text{flow incompressible} \end{array} \right.$

Ex.

Consider particle beam  radius R_0, focused to R_1

Focus? (Hint $\rightarrow$ only need know Hamiltonian system)



transverse momentum dispersion Δp_0

$\int$ 

Key $\rightarrow$ phase space volume constant (i.e. phase space flow incompressible)

$$\Rightarrow (\rho)\pi R_0^2 \, \pi \Delta p_0^2 = \pi R_1^2 \pi (\Delta p_1)^2 (\rho)$$

$$\therefore \quad \Delta p_1 = \frac{R_0}{R_1} \Delta p_0$$

i.e. momentum dispersion increases to compensate reduction in R

$\Rightarrow$ focusing inefficient

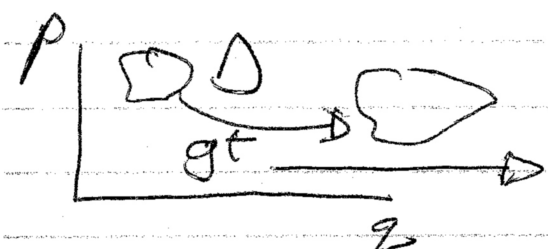
Liouville Thm. – cont'd.

Hereafter, take $H = H(\underline{p}, \underline{q})$.

Define phase flow: g^t transformation s/t

$$\underline{p}(0), \underline{q}(0) \rightarrow \underline{p}(t), \underline{q}(t) \text{ along Hamiltonian trajectories}$$

Now,



g^t is evolution operator

$$\boxed{\text{Volume } g^t D = \text{Volume } D} \rightarrow \text{obvious equivalent } dD/dt = 0$$

to show: Hamiltons eqns. autonomous

$\underline{\underline{\text{i.e.}}}$ $\quad \dot{\underline{x}} = \underline{f}(\underline{x}) \quad \leftarrow$ can be written

$$\Rightarrow \quad g^t(x) = \underline{x} + \underline{f}(\underline{x})t + o(t^2)$$

Jacobian transformation

$$V(t) = \int_{D(0)} d\underline{x} \left| \frac{\partial \underline{x}'}{\partial \underline{x}} \right| = \int_{D(0)} d\underline{x} \det \frac{\partial g^t(\underline{x})}{\partial \underline{x}}$$

$$\frac{\partial g^t(\underline{x})}{\partial \underline{x}} = \underline{\underline{I}} + \frac{\partial \underline{f}}{\partial \underline{x}} t + o(t^2)$$

using identity: $\det(\underline{\underline{I}} + \underline{\underline{A}}t) = 1 + t\,\mathrm{tr}\,\underline{\underline{A}} + o(t^2)$

$$V(t) = \int_{\Omega(0)} \left[1 + t\,\mathrm{tr}\left[\frac{\partial \underline{f}}{\partial \underline{x}}\right] + o(t^2)\right] d^3x$$

$$\mathrm{tr}\,\frac{\partial \underline{f}}{\partial \underline{x}} = \underline{\nabla}\cdot\underline{f}, \quad \text{but } \underline{\nabla}\cdot\underline{f} = \underline{\nabla}\cdot\underline{V}_{\Gamma} = 0!$$

i.e. f is phase space flow velocity
phase space $\dot{\underline{x}} = \underline{f}(\underline{x})$

so $V(t) = V(0)$

$\Rightarrow$ 1) no attractors in Hamiltonian mechanics, i.e.
$\Rightarrow$ no asymptotically stable positions, cycles,

2) Poincare Recurrence Thm. – fundamental to ergodic theory.

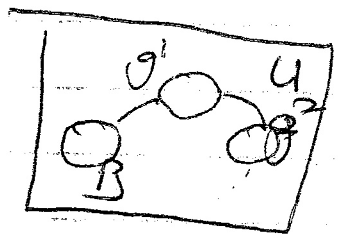
– "what goes around, comes around, arbitrarily closely", in H-mech.

i.e. $\mathcal{U} \equiv$ system universe, – bounded
g Hamiltonian $\Rightarrow$ volume preserving

$\exists$ for any $\underline{x}$, can define $B(\underline{x}, \epsilon)$

ball at $\underline{x}$, radius ϵ

then $\exists\, \underline{x}' \in B(\underline{x})$ s/t $g^n(\underline{x}') \in B(\underline{x})$

i.e.  consider $g^n(B)$.

if disjoint, $\cup g^n \Rightarrow \infty$ $\Rightarrow$ infinite volume $\Downarrow$ contradiction

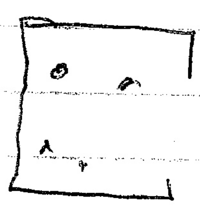
so $g^k(B) \cap g^\ell(B) \neq \emptyset$

$\Rightarrow$ $g^{k-\ell}(B) \cap B \neq \emptyset$

$\therefore$ $\exists\, \underline{x}' \in g^{k-\ell}(B) \cap B$

$\Rightarrow$ $\underline{X}'$ arbitrarily close to $\underline{x}$. QED

Implications

–  $\longrightarrow$ out

go back $\Leftrightarrow$ but may take a while . . .

– mapping circle onto itself brings n^{th} image of pt. orbit close.

if  $\dot{\varphi}_1 = \alpha_1$, $\dot{\varphi}_2 = \alpha_2$ $\quad g^t(\varphi_1, \varphi_2) \to (\varphi_1 + \alpha_1 t, \varphi_2 + \alpha_2 t)$

α_1/α_2 irrational $\Rightarrow$ winding fills torus.